MODULE 9: INTRODUCTION TO CONSTANT RATES

We've studied rational numbers in the last five modules. Rational numbers occur when we deal with *rates*. Recall that a rate is indicated by a phrase consisting of *two nouns*, separated by the word *"per"*.

More specifically:

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*********************************************
*                                           *
*   The Vocabulary of Rates                 *
*       If A and B denote any two nouns,    *
*   then "A's per B" is called the rate of  *
*   change of A with respect to B.          *
*                                           *
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Example 1

A 4 pound package of ground beef costs $8. What is the cost of 1 pound?

Answer: *$2*

The label of our answer will be "dollars per pound". In terms of arithmetic this means that we must divide the number of dollars (8) by the number of pounds (4). We get:

8 dollars ÷ 4 pounds = 2 *dollars per pound*

$2 per pound is a rate of change. It tells us that the amount we have to pay changes by $2 for each pound we buy.

We've discussed this in earlier modules. Whenever "per" appears between two nouns, it can be replaced by the division sign. In fact we often write "dollars per pound" as:

$$\frac{\textit{dollars}}{\textit{pound}}$$

Note the Importance of Order

If we had written 4 pounds ÷ 8 dollars, the answer would have been ½ pound per dollar. In other words "dollars per pound" and "pounds per dollar" are reciprocals. That is, $2 per pound and ½ pound per dollar are two different ways of saying the same thing.

If we use the language of common fraction, both of the following are correct:

$$\frac{\textit{2 dollars}}{\textit{1 pound}} \textit{ and } \frac{\textit{1 dollar}}{\textit{2 pounds}}$$

In the first case the ratio is 2 while in the second it is ½.

Example 1 is an illustration of what we call a *constant* rate of change. It assumes that the price per pound doesn't change. You are already familiar with many examples of constant rate of change. For instance, there are always 12 inches per foot; there are always 100 cents per dollar; there are always 12 per dozen.

Rather than say there are 12 inches per foot, we say that there are 12 inches in a foot. So you might have to be on the lookout for words that mean the same thing as "per".

On the other hand, there are many examples of rates of change that are not constant. For instance, when you're driving the number of miles you drive in one hour is usually not the same as the number of miles you drive in another hour. When you write checks to pay for purchases, the amount per check is often not the same.

In an hour on the super highway you expect to drive further than in an hour of driving through crowded streets during rush hour.

That is, two purchases are rarely exactly the same amounts.

In Module 11 we'll deal with rates that are not constant. But for now we'll restrict our attention to constant rates of change.

Example 2

There are 12 inches per foot. How many inches are there in 7 feet?

Answer: 84

Here we have a constant rate of change. Everytime a length changes by 1 foot it also changes by 12 inches. That is, since there are 12 inches in one foot, there will be 12 inches seven times in 7 feet. More formally:

7 feet = 7 X 1 foot

That is, we have 1 foot, 7 times.

= 7 X (12 inches)

We've replaced "1 foot" by its synonym "12 inches"

$= (7 \times 12)$ inches

$= 84$ inches.

In Module 2 we learned that 7 X 12 = 84. In this modul we have to know that we wan to multiply 7 by 12.

Do you see how we handled the fact that there were always 12 inches per foot? To find the number of inches we took the number of feet and multiplied it by 12. It is as if we had a computer program that said:

1. *Enter a number (the number of feet)*
2. *Multiply by 12 (the number of inches per foot)*
3. *Write the answer (the number of inches)*

So in terms of Example 2, we entered 7, multiplied by 12, and the answer was the number of inches (in 7 feet

Examples 1 and 2 involved whole numbers. This need not be the case when we deal with rates.

Example 3

How many inches are there in $4\frac{2}{3}$ feet?

Answer: 56

We again "enter" the number of feet ($4\frac{2}{3}$) and multiply by 12. We get:

$$4\frac{2}{3} \times 12 = \frac{14}{3} \times 12$$
$$= \frac{14}{3} \times \frac{12}{1}$$
$$= \frac{14}{\cancel{3}_{1}} \times \frac{\cancel{12}^{4}}{1}$$
$$= \frac{56}{1}$$
$$= 56$$

In this module we have to know that we are going to multiply the number of feet by 12. Once we know this we use the computational skills we learned in the earlier modules. But if we don't know what to do with them, the computational skills taught in those modules are essentially useles

So even though the number of feet was a mixed number, we still multiplied the number of feet by the number of inches per foot (the constant rate) to get the total number of inches.

Based on our discussion in Examples 2 and 3, perhaps you can see the following pattern emerging.

A Recipe

If the rate of change of A with respect to B is constant, there is an easy way to find the change in A given the change in B. Namely:

(1) Take the change in B.

(2) Multiply it by the rate of change of A with respect to B.

(3) The product is the corresponding change in A.

See the importance of having a constant rate? For example if we drive at different speeds for 5 hours, what speed do we multiply the 5 hours by to find how far we went? But if we knew that we always drove at 30 miles per hours, then we'd know that we went (5 X 30) or 150 miles.

Example 4

If there are 60 minutes in an hour, how many minutes are there in 2.4 hours?

Answer: 144

In this example think of A as being minutes and B as being hours. So the constant rate of change of A with respect to B is 60.

The change in B is 2.4 hours.

Hence we multiply 2.4 by 60 to get 144.

0.1 hours is 1/10 of 60 minutes or 6 minutes. Hence 0.4 hours in 24 minutes. So 2.4 hours in a convenient decimal way of saying 2 hours and 24 minutes. In fact if we were dealing with 2 hours and 24 minutes, we'd rewrite it as 2.4 hours if we wanted to use a calculator.

Rough Check

2 hrs < 2.4 hrs < 3 hrs.

In 2 hours there are 2 X 60 or 120 minutes, and in 3 hours there are 3 X 60 or 180 minutes. Hence in 2.4 hours there must be more than 120 minutes but less than 180 minutes. 144 minutes is in this range.

To think of 2.4 hours as the change in the number of hours, think of the start of the problem being at 0 hours.

When we deal abstractly with A's and B's it is easy to confuse the rate of change of A with respect to B with the rate of change of B with respect to A. If we use the language of common fractions, there is a relatively easy way to eliminate this problem.

To see how this is done, let's first explore some preliminary results.

Example 5

How much is 15 inches ÷ 5 inches?

Answer: 3 (not 3 inches!)

Recall that

15 inches ÷ 5 inches = _____

means the same thing as

15 inches = 5 inches X _____

The definition of divisior never changes in this course. $f \div s = t$ always means the same as $f = s \times t$

As illustrated below, we need three 5 inch lengths to make one 15 inch length.

That is, we need a 5 inch length 3 times to equal a 15 inch length.

|← 5 in →|← 5 in →|← 5 in →|

|← 15 inches →|

The point is this. Suppose we rewrote the division problem in Example 5 in the language of common fractions. We'd have:

$$\frac{15 \text{ inches}}{5 \text{ inches}} = 3$$

In this form, we see that we may cancel common denominations *in the same way as we cancel common factors.*

That is, it is as if we cancelled "inches" to get:

$$\frac{15\ \cancel{\text{inches}}}{5\ \cancel{\text{inches}}} = \frac{15}{5} = 3$$

Example 6

How much is 15 inches ÷ 5?

Answer: 3 inches (not 3)

Make sure you see the subt difference between Example 6 and 7.

15 inches ÷ 5 = ______

means the same as:

15 inches = 5 X ______

That is, 5 of what length is equal to a length of 15 inches. Clearly it takes

five 3 inch lengths to equal one 15 inch length.

If we rewrote the example in the language of common fractions, we'd have:

$$\frac{15 \text{ inches}}{3} = 5 \textbf{ inches}$$

In this case the "inches" doesn't cancel because it is not common to both numerator and denominator.

"inches" occurs only in the numerator so we can't cancel it.

Example 7

Simplify:

$$\frac{5 \text{ dimes}}{2 \text{ quarters}} \times \frac{4 \text{ quarters}}{1 \text{ dollar}}$$

Since we can cancel common denominations as well as common factors, we cancel "quarters" from numerator and denominator to get:

Answer: $\frac{10 \textit{ dimes}}{1 \textit{ dollar}}$ *or 10 dimes per dollar*

$$\frac{5 \text{ dimes} \times \overset{2}{\cancel{4}} \cancel{\text{quarters}}}{\underset{1}{\cancel{2}} \cancel{\text{quarters}} \times 1 \text{ dollar}} =$$

$$\frac{5 \text{ dimes} \times 2}{1 \times 1 \text{ dollar}} =$$

$\frac{10 \text{ dimes}}{1 \text{ dollar}}$ or 10 dimes per dollar

That is a/b may be read as a's per b.

While what we did in Example 7 may look very mechanical it helps us solve a very important rate problem. What the problem actually tells us is that if there are 5 dimes per 2 quarters and 4 quarters per dollar, then *at that rate,* there are 10 dimes per dollar; a fact that is easily checked with reality. With this new notation in mind, let's revisit Example 2. We want to know how many inches there are in 7 feet.

$\frac{5 \textit{ dimes}}{2 \textit{ quarters}}$ *means there are 5 dimes per 2 quarters and* $\frac{4 \textit{ quarters}}{1 \textit{ dollar}}$ *means that there are 4 quarters per dollar.*

THE PROBLEM:

To find the number of inches in 7 feet.

This is a rather easy problem but it serves as a gateway to more complicated problems.

Step 1:

Write the problem in terms of fill in the blank:

$$7 \text{ feet} = \underline{\qquad} \text{ inches}$$

Step 2:

Rewrite the left side to look more like a common fraction:

$$\frac{7 \text{ feet}}{1} = \underline{\qquad} \text{ inches}$$

Our strategy is to take advantage of cancelling common denominations from numerator and denominator, and to convert the label on the left side to the label on the right side.

Step 3:

The left side is in "feet" and we want the answer (the right side) in "inches. So start a new factor with "feet" in the denominator:

$$\frac{7 \text{ feet}}{1} \times \frac{\quad}{\text{feet}} = \underline{\qquad} \text{ inches}$$

In this way "feet" in the denominator cancels with "feet" in the numerator.

Step 4:

Since we want the answer to be in inches, label the numerator of the new fraction "inches":

$$\frac{7 \text{ feet}}{1} \times \frac{\text{inches}}{\text{feet}} = \underline{\qquad} \text{ inches}$$

As a check, the "feet" on th left side cancel and the label becomes "inches", which is what we want.

Step 5:

The constant rate of 12 inches per 1 foot tells us to write 12 as the adjective that modifies "inches" and 1 as the adjective that modifies "feet".

$$\frac{7 \text{ feet}}{1} \times \frac{12 \text{ inches}}{1 \text{ foot}} = \underline{\qquad} \text{ inches}$$

In the context of cancelling common denominations, we do not distinguish between "feet" and "foot". That is, singular or plural the denomination is the same.

Step 6:

Do the resulting arithmetic to get:

$$\frac{7 \cancel{\text{feet}} \times 12 \text{ inches}}{1 \times 1 \cancel{\text{foot}}} = \frac{84 \text{ inches}}{1}$$

$$= 84 \text{ inches}$$

Using these 6 steps reduces our need to keep track of whether we're dealing with A's per B or with B's per A.

Example 8

How many feet are there in 192 inches?

Answer: 18

In terms of fill in the blank, we have:

192 inches = ______ feet.

So think of 192 inches as $\frac{192 \text{ inches}}{1}$.

In this form we have inches (in the numerator) and we want the answer in feet. So we want to multiply by a fraction whose denominator is inches and whose numerator is feet. So using the relationship that there are still 12 inches per foot, we have:

$$\frac{192 \text{ inches}}{1} \times \frac{1 \text{ foot}}{12 \text{ inches}} = \frac{192 \cancel{\text{inches}} \times 1 \text{ foot}}{1 \times 12 \cancel{\text{inches}}}$$

$$= \frac{192 \text{ feet}}{12}$$

$$= 18 \text{ feet.}$$

$\frac{12 \text{ inches}}{1 \text{ foot}}$ and $\frac{1 \text{ foot}}{12 \text{ inches}}$ mean the same thing; but whether "inches" is in the numerator or denominator determines whether we multiply by 12 or divide by 12.

Note:

In most cases we know that we're either going to multiply by 12 or divide by 12, but we aren't always sure of which. A good devise is to remember that the bigger the unit the less the adjective. That is, since a foot is more than an inch, a given length has more inches than feet. So in going from inches to feet we divide by 12.

The nice thing about the common fraction notation is that the position of the adjectives in the numerator or denominator tells us whether we should multiply or divide.

More generally, in going from the bigger unit to the smaller we multiply but we divide when we go from the smaller unit to the greater.

Let's look at a few other applications.

Example 9

A football league has 32 teams and each team has 45 players. How many players are there in the league?

Answer: 1,440

Method 1

Since each team has 45 players and there are 32 teams in the league, we have 45 players, 32 times. Hence there are 45 X 32 or 1,440 players.

Make sure that you understand that this is a constı rate problem. That is, eaı team changes the number of players by 45.

Method 2

In terms of common fraction notation we have:

$$\frac{32 \text{ teams}}{1} \times \frac{45 \text{ players}}{1 \text{ team}} =$$

$$\frac{32 \cancel{\text{ teams}} \times 45 \text{ players}}{1 \times 1 \cancel{\text{ team}}} =$$

$$\frac{1{,}440 \text{ players}}{1} =$$

$$1{,}440 \text{ players}$$

See what we're doing? We'ı starting with 32 teams and finding a way to cancel "teams" and bring in "players".

Many different situations are solved by the same mathematical computations. For instance:

Example 10

A clothing store owner buys 32 coats at \$45 per coat. What is the total price that he pays for the coat?

Answer: \$1,440

He has to pay \$45, 32 times. That is, he pays \$45 X 32 or \$1,440 for the coats.

In terms of common fractions:

$$\frac{32 \text{ coats}}{1} \times \frac{45 \text{ dollars}}{1 \text{ coat}} = \frac{32 \cancel{\text{ coats}} \times 45 \text{ dollars}}{1 \times 1 \cancel{\text{ coat}}}$$

$$= \frac{1{,}440 \text{ dollars}}{1}$$

$$= 1{,}440 \text{ dollars}$$

Of course we could also have written:

$$\frac{45 \text{ dollars}}{1 \text{ coat}} \times 32 \text{ coats} =$$

$$\frac{45 \text{ dollars}}{1 \cancel{\text{ coat}}} \times \frac{32 \cancel{\text{ coats}}}{1} =$$

1,440 dollars

While Examples 9 and 10 describe different situations, they are mathematically equivalent. And in a sense the different types of problems that are solved this way are virtually endless.

Example 11

At a rate of 45 miles per gallon, how far will an economy car travel on 32 gallons of gas?

Answer: 1,440 miles

For each gallon of gas the car travels 45 miles. Hence on 32 gallons of gas, the car will travel 45 miles, 32 times--that is, 32 X 45 or 1,440 miles.

In terms of common fractions:

$$\frac{32 \cancel{\text{gallons}}}{1} \times \frac{45 \text{ miles}}{1 \cancel{\text{gallon}}} = 1{,}440 \text{ miles}$$

In one sense Example 11 is different from Examples 9 and 10. It is common for all teams to have the same number of players and it is common for all coats to cost the same amount. But in general, a car will not get the same number of miles on each gallon of gas. For this reason we often talk about a car getting an average *of 45 miles per gallon. In Module 11 we'll say more about average rate of change.*

It's important to keep track of labels as we do these problems. We can get into trouble if we simply multiply indiscriminately.

Example 12

If a car travels at a rate of 45 miles per hour, how far will it travel in 20 minutes?

Answer: 15 miles

Resist the temptation of multiplying 45 by 20. Based on what we did so far, we should recognize that we want to multiply "miles per hour" by "hours" to get "miles". That is:

$$\frac{\text{miles}}{\cancel{\text{hour}}} \times \cancel{\text{hours}} = \text{miles}$$

Hint for guarding against multiplying 45 by 20:

20 minutes is less than an hour. In an hour the car goes 45 miles. So in 20 minutes it must travel less than 45 miles. Therefore "900 miles" is actually a preposterous answer!

The problem is that we're given the time in minutes. Recalling that a minute is $\frac{1}{60}$ of an hour, 20 minutes is $\frac{20}{60}$ or $\frac{1}{3}$ hour.

Hence:

$$\frac{45 \text{ miles}}{1 \text{ hour}} \times \frac{1}{3} \text{ hours} = \frac{45 \text{ miles} \times 1 \cancel{\text{hours}}}{1 \cancel{\text{hour}} \times 3}$$

$$= \frac{45 \text{ miles}}{3}$$

$$= 15 \text{ miles}$$

In terms of common fractions, we want to get from minutes to hours to miles, so we write:

$$\frac{20 \cancel{\text{minutes}}}{1} \times \frac{1 \cancel{\text{hour}}}{60 \cancel{\text{minutes}}} \times \frac{45 \text{ miles}}{1 \cancel{\text{hour}}} =$$

$$\frac{20 \times 1 \times 45 \text{ miles}}{1 \times 60 \times 1} =$$

$$\frac{900 \text{ miles}}{60} =$$

$$15 \text{ miles}$$

By multiplying first by $\frac{\text{hours}}{\text{minutes}}$ we cancel "minutes" and introduce "hours". Then when we multiply by $\frac{\text{miles}}{\text{hours}}$ we cancel "hours" and get "miles".

An Alternative Method:

45 miles per hour is the same as 45 miles per 60 minutes or:

$$\frac{45 \text{ miles}}{60 \text{ minutes}} =$$

$$\frac{3 \text{ miles}}{4 \text{ minutes}} =$$

$$\frac{3}{4} \text{ miles per minute.}$$

3 miles per 4 minutes is the same as 3/4 miles per minute That is:

$$\frac{3 \text{ miles}}{4 \text{ minutes}} = \frac{3}{4} \ \frac{\text{miles}}{\text{minutes}}$$

$$= \frac{3}{4} \text{ miles per minut}$$

And if we go $\frac{3}{4}$ of a mile every minute then in 20 minutes, we go $\frac{3}{4}$ of a mile 20 times or:

$$\frac{3}{4} \times 20 \text{ or } 15 \text{ miles}$$

Any method that helps you understand why the answer is 15 miles is fine. Don't get locked into a method. Try to understand what's going on in the problem.

Sometimes the same situation can be stated with a different emphasis. For example in Example 12, we wanted to see how far the car went in a given amount of time. We might with equal logic have wanted to know how long it took to travel a given number of miles.

Example 13

At a rate of 45 miles per hour, how long will it take for a car to travel 81 miles?

Answer: 1 hour and 48 minutes.

We have:

81 miles = ______ hours

Plausibility Check

In 2 hours the car would go 2 X 45 or 90 miles. So it should take a "little less" than 2 hours to go 81 miles. Hence 1 hour and 48 minutes is a reasonable answer.

So we want to get rid of "miles" and bring in "hours". Since "miles" appears in the numerator we want to have it in the denominator,

Therefore:

$$\frac{81 \text{ miles}}{1} \times \frac{1 \text{ hour}}{45 \text{ miles}} = \frac{81 \text{ miles} \times 1 \text{ hour}}{1 \times 45 \text{ miles}}$$

$$= \frac{81 \cancel{\text{miles}} \times 1 \text{ hour}}{1 \times 45 \cancel{\text{miles}}}$$

$$= \frac{81 \text{ hours}}{45}$$

$$= \frac{81}{45} \text{ hours}$$

$$= \frac{9}{5} \text{ hours}$$

$$= 1\frac{4}{5} \text{ hours}$$

$$= 1 \text{ hour and } 48 \text{ minutes}$$

We don't have to memorize that we must divide by 45. The fact that 45 wound up in the denominator tells us to divide by 45.

1/5 of an hour is 1/5 of 60 minutes or 12 minutes. Hence 4/5 of an hour is 4 X 12 minutes or 48 minutes.

Check

1 hour and 48 minutes is 60 minutes + 48 minutes or 108 minutes. The car goes 3/4 of a mile in one minute. So in 108 minutes it goes 3/4 of a mile 108 times:

$$\frac{3}{4} \times 108 = \frac{3 \times 108}{4}$$

$$= \frac{324}{4}$$

$$= 81 \text{ (miles)}$$

While you can travel for $\frac{3}{4}$ of a mile, you can't, for all practical purposes, buy $\frac{3}{4}$ of a coat.

Example 14

At \$47 dollars per coat, how many coats can a merchant buy for \$1,000?

Answer: 21 (with \$13 left)

In this case we want to see how many 47's there are in 1,000. By division we get:

# of coats X \$47 = cost

__________ X \$47 = \$1,000

$$\begin{array}{r} 21 \text{ R13} \\ 47\overline{)1000} \\ -94 \\ \hline 60 \\ -47 \\ \hline 13 \end{array}$$

Technically the answer is $21\frac{13}{47}$ which means he can buy 21 coats and have left \$13 of the \$47 another coat would cost.

So in this example we multiply 47 X 21 to get \$987 and then we have to subtract \$987 from \$1,000 to get the amount of money the merchant has left.

The formula is

(# of coats) X \$47 = Total Cost.

So if we know the number of coats, we multiply by \$47 to find the total cost; but if we know the total cost, we divide by \$47 to find the number of coats we can buy.

That is, ____ X \$47 = \$1,000 looks like a multiplication problem but it's really a division problem

Again, however, if we use the common fraction notation, whether we multiply or divide takes care of itself. Namely:

$$\frac{1{,}000 \text{ dollars}}{1} \times \frac{1 \text{ coat}}{47 \text{ dollars}} =$$

$$\frac{1{,}000 \cancel{\text{dollars}} \times 1 \text{ coat}}{1 \times 47 \cancel{\text{dollars}}} =$$

$$\frac{1{,}000 \text{ coats}}{47} = \left(\frac{1{,}000}{47}\right) \text{ coats}$$

We make sure "dollars" are in the denominator so that we can cancel it with the "dollars" in the numerator.

In many real-life situations we are required to subtract in order to find the change in a quantity. For example, many times you find how far you had to travel by recording your odometer reading at the beginning of the trip and subtracting this reading from the odometer reading at the end of the trip. Let's illustrate this by looking at one problem in two different ways.

The odometer is that part of the speedometer that records the number of miles the car has been driven.

Example 15

A person travels 306 miles in 6 hours. What was his average speed for the trip?

Answer: 51 miles per hour

When we deal with miles and hours, speed is measured in miles per hour. Recalling that miles per hour means miles ÷ hours, we have:

306 miles ÷ 6 hours = 51 miles per hour

In the language of common fractions:

$$\frac{306 \text{ miles}}{6 \text{ hours}} = 51 \, \frac{\text{miles}}{\text{hour}}$$

Note:

We call the answer the average speed because it is unrealistic to assume that we travelled at a constant rate of 51 miles per hour. The point is that any trip in which we went 306 miles in 6 hours would have an average speed of 51 miles per hour.

From a different point of view, if we knew that the speed of the car were constant, then the speed would have been exactly 51 miles per hour.

If we preferred to use common fractions, we could have written:

1 hour = ____ miles

"miles per hour" is the same as asking how many miles we went in 1 hour.

$$\frac{1 \text{ hour}}{1} \times \frac{306 \text{ miles}}{6 \text{ hours}} =$$

$$\frac{1 \cancel{\text{hour}} \times 306 \text{ miles}}{1 \times 6 \cancel{\text{hours}}} =$$

$$\frac{306 \text{ miles}}{6 \text{ hours}} =$$

$$51 \, \frac{\text{miles}}{\text{hour}}$$

Recall that $\frac{\text{miles}}{\text{hour}}$ *is read as "miles per hour"*

As a plausibility argument about our answer to Example 15 notice that if we had only gone 300 miles in 6 hours, our average speed would have been 50 miles per hour. The fact that we went 306 miles in 6 hours indicates that the average speed should be a bit more than 50 miles per hour--and 51 miles per hour certainly seems reasonable in this respect.

A more important question might have been how the driver could be sure that the trip was 306 miles. So let's see how he could use the odometer for this.

<u>Example 16</u>

At the start of a trip the car's odometer reads 021547 miles. 6 hours later the odometer reads 021853 miles. What was the average speed of the car during these 6 hours?

Answer: 51 miles per hou

We want the rate of change of distance with respect to time. The time changed by 6 hours. At the start of the 6 hour trip, the car had been driven 021547 miles. At the end of the trip it had been driven 021653 miles. So during the trip the car had been driven:

$$\begin{array}{rl} & 021853 \text{ miles} \\ - & \underline{021547 \text{ miles}} \\ & 000306 \text{ miles} \end{array}$$

That is, we have to add 3(to 21547 to get 21853

So we went 306 miles in 6 hours, which presents us with exactly the same problem we solved in Example 15.

Reading the odometer is also the usual way for determining gas milage.

Example 17

Under the conditions of Example 16, the driver starts the trip with a full tank of gas. When the 6 hour trip is over, he finds that it takes 12 gallons of gas to fill the tank. What was the average miles per gallon (mpg) for the trip?

Answer: 25.5 mpg

To find the average miles per gallon, we want the number of miles (which is 306) divided by the number of gallons of gas that were used in the trip (12). We get:

So if we hadn't already subtracted the starting and ending odometer readings, we'd do so now.

$$\begin{array}{r} 25.5 \\ 12\overline{)306.0} \\ -24 \\ \hline 66 \\ -60 \\ \hline 6\,0 \\ -6\,0 \\ \hline 0 \end{array}$$

We could also write the answer as $25\frac{1}{2}$ mpg

It is worth noticing by comparing Examples 16 and 17 why we say the rate of change of one quantity <u>with respect to</u> another quantity. In both of these examples, we found a rate of change of mileage, but in Example 16 it was with respect to time while in Example 17 it was with respect to fuel consumption.

In all of these problems the key relationship has been that if the number of B's per A is constant, then:

$$\text{Number of A's} \times \frac{\text{Number of B's}}{\text{per A}} = \text{Number of B's}$$

That is:

$$\frac{\#\text{ of A's}}{1} \times \frac{\#\text{ of B's}}{1\text{ A}} = \#\text{ of B's}$$

or in alternative form:

Number of B's ÷ Number of A's

=

(Average) Number of B's per A

Example 18

At \$7.95 per gallon, what is the cost of 16 gallons of paint?

Answer: \$127.20

Each gallon costs \$7.95, so the price of 16 gallons is \$7.95 X 16 or \$127.20.

This is where a calculator starts to become helpful. But remember that the calculator can't help you if you don't know you're supposed to multiply \$7.95 by 16.

In other words:

$$\frac{7.95 \text{ dollars}}{1 \cancel{\text{gallon}}} \times 16 \cancel{\text{gallons}} =$$

(7.95 X 16) dollars.

Example 19

You can buy $2\frac{3}{4}$ pounds of meat for \$4.95
At this rate, what was the price of the meat in dollars per pound?

Answer: \$1.80

The label "dollars per pound" tells us that we want to divide dollars (4.95) by pounds ($2\frac{3}{4}$).

We get:

4.95 dollars $\div 2\frac{3}{4}$ pounds =

Remember that dollars ÷ pounds means dollars per pound.

$(4.95 \div 2\frac{3}{4})$ dollars per pound =

$\frac{495}{100} \div \frac{11}{4}$ dollars per pound =

Using the calculator we might prefer to rewrite $2\frac{3}{4}$ as 2.75 and do the problem as 4.95 ÷ 2.75

$\frac{495}{100} \times \frac{4}{11}$ dollars per pound =

$\frac{495 \times 4}{100 \times 11}$ dollars per pound =

$\frac{\overset{\overset{9}{\cancel{99}}}{\cancel{495}}}{\underset{\underset{5}{\cancel{25}}}{\cancel{100}}} \times \frac{\overset{1}{\cancel{4}}}{\underset{1}{\cancel{11}}}$ dollars per pound =

We've elected to convert all ratios to common fractions and use the results of Module 5.

$\frac{9}{5}$ dollars per pound =

$$\frac{9}{5} = 5\overline{)9.00} = 1.80 \qquad \begin{array}{r} 1.8\,0 \\ 5\overline{)9.0\,0} \\ -5 \\ \hline 4\,0 \\ -4\,0 \\ \hline 0 \end{array}$$

\$1.80 per pound

Note that we may also read $\frac{9}{5}$ dollars per pound as $\frac{9 \text{ dollars}}{5 \text{ pounds}}$ or 9 dollars for 5 pounds.

Note:

If you had divided $2\frac{3}{4}$ pounds by 4.95 dollars, the label would have been pounds per dollar and the answer would have been $\frac{5}{9}$ pounds for each (per) dollar. While $\frac{5}{9}$ and $\frac{9}{5}$ are reciprocals,

$$\frac{5 \text{ pounds}}{9 \text{ dollars}} \text{ and } \frac{9 \text{ dollars}}{5 \text{ pounds}}$$

say exactly the same thing. In a rounded-off form, \$2 per pound means the same thing as ½ pound per \$1.

To decide which way is correct in Example 19, notice that since 2.75 pounds cost more than \$2.75, the cost per pound has to be more than \$1. So \$5/9 can't be right. Moreover, by rounding off we're getting around 3 pounds for \$5. This tells us that the answer is in the vicinity of \$1.70 per pound.

Once we know the price per pound we can find the price of any number of pounds, and we can also find how many pounds we can buy for a given number of dollars.

Example 20

You can buy $2\frac{3}{4}$ pounds of meat for \$4.95. At this rate, how much would 90 pounds cost?

Answer: \$162

In Example 19 we found that the cost per pound was \$1.80. So all we have to do is multiply the number of pounds we want to buy (90) by \$1.80 and we get \$162.

That is:

$$\frac{90 \text{ pounds}}{1} \times \frac{1.80 \text{ dollars}}{1 \text{ pound}} =$$

$$\frac{90 \cancel{\text{pounds}} \times 1.80 \text{ dollars}}{1 \times 1 \cancel{\text{pound}}} =$$

$$\frac{162 \text{ dollars}}{1} = \$162.$$

If we hadn't already done Example 19, we could have written:

$$\frac{90 \text{ pounds}}{1} \times \frac{4.95 \text{ dollars}}{2\frac{3}{4} \text{ pounds}}$$

In this sense we would do Example 19 now.

Note that we've done this problem on the basis of constant rate (\$1.80 per pound). Keep in mind, however, that to encourage larger purchases, merchants will usually offer lower rates for larger orders.

For example, the merchant might be willing to sell you the 90 pounds for \$140 and in this case you'd be paying less than \$1.80 per pound.

Example 21

You can buy $2\frac{3}{4}$ pounds of meat for \$4.95. At this rate how many pounds can you buy for \$90?

Answer: 50

Every time we buy a pound we spend \$1.80. So we can buy as many pounds as there are \$1.80's in \$90. Therefore we need only divide \$90 by \$1.80 to get:

$$\frac{\$9.00}{\$1.80} = \frac{\$900}{\$180} = 180\overline{)900}\ \ 50 \quad -90 \quad 0$$

Again, in terms of common fractions:

$$\frac{90 \text{ dollars}}{1} \times \frac{1 \text{ pound}}{1.80 \text{ dollars}} =$$

$$\frac{90 \cancel{\text{dollars}} \times 1 \text{ pound}}{1 \times 1,80 \cancel{\text{dollars}}} =$$

$$\frac{90 \text{ pounds}}{1.80} =$$

$$\frac{9000}{180} \text{ pounds} = 50 \text{ pounds}$$

It is easy to confuse Examples 20 and 21. That is 90 dollars and 90 pounds use the same adjective. The following thought process might be helpful. Since a pound costs more than \$1, 90 pounds must cost more than \$90. Hence \$50 would not be a reasonable answer in Example 20. Conversely since you get less than a pound per dollar, you get less than 90 pounds for \$90. So 162 wouldn't be a reasonable answer to Example 21

1.80 always modifies dollars So the fact that dollars is in the denominator tells us that we're dividing by 1.80

Other places that constant rates are of interest to consumers include taxation. Suppose there's a 5% sales tax in your town. This means that for every dollar something costs, you have to pay \$1.05 to buy it. In this case the constant sales tax rate is \$1.05 total price per \$1 marked price.

5% means 5 per hundred. So the tax would be 5¢ per 100¢ Hence the total cost would be \$1.05 for each \$1 the object costs.

The "Program" would be:

1. *Enter the "Marked Price" in dollars*
2. *Multiply by 1.05*
3. *The answer is the cost including the tax.*

Do you understand "marked price"? If the price you se marked on an object is \$3, you must pay an additional 5¢ three times (once for each dollar) or 15¢. Hence the price including the tax would be \$3.15

Marked Price in Dollars	*X*	*1.05*	*=*	*Total Cost including the Tax*

Example 22

The marked price of a radio is \$87. What is the price including the sales tax if the sales tax is 5%?

Answer: \$91.35

For each dollar the price is marked we have to pay \$1.05 to take care of the tax. Hence if the radio is marked \$87 we must pay \$1.05 87 times. This leads to:

87 X \$1.05 = \$91.35

Note:

If you multiplied \$87 by 5% and got \$4.35, you found the tax. To find the total price the tax has to be added to the marked price. This would give us \$87 + \$4.35 or \$91.35 as the total price including the tax. By multiplying \$87 by 1.05 we combine the two steps into one.

You may prefer to think in terms of a table:

Marked Price	*Total Price*
\$1	*\$1.05*
\$2	*\$2.10*
\$3	*\$3.15*
\$4	*\$4.20*
⋮	⋮
\$87	*\$91.35*

The first column is the multiples of 1 and the second is the multiples of 1.05 So, for example, since \$87 is the 87th multiple of 1. \$91.35 is the 87th multiple of \$1.05

Another way to view this is that the object is 100% of the marked price and we pay an additional 5% of the marked price for the tax. So altogether we pay 105% of the marked price.

Notice that the 5% is a rate not a total amount. That is, it tells us that we add 5¢ to each \$1. Hence the more dollars we spend the more the tax is, even though the rate (5%) never changes.

Example 23

You pay \$8,900 for a new car. In addition there is a 5% sales tax. How much money will the sales tax cost you?

Answer: \$445

You're paying a tax of \$0.05 for each \$1 of the price of the car. Hence your tax is \$0.05, 8,900 times or \$0.05 X 8,900 or \$445.

The rate is still 5¢ for each dollar, but the cost is a "lot of" dollars.

5% may not seem like that much but at a rate of 5¢ per \$1, you will have to pay \$445 per \$8,900. If you don't anticipate this \$445 tax, you can cause yourself a major financial problem.

Note:

If you had multiplied \$8,900 by 1.05 you would have found the total cost of the car including the tax. This amount is \$9,345. In this case, you'd then subtract the marked price (\$8,900) to get the amount of the tax.

Many times a consumer doesn't seem to care too much whether the tax is 5% or 6% since there is only a difference of 1 part per 100. But on a "big ticket" item this can be a major difference.

Expensive items are often referred to as "big ticket" items.

Example 24

How much would the tax on the car of Example 23 have been had the sales tax rate been 6%?

Answer: \$534

This time the tax is costing you 6¢ for each dollar the car is marked. Hence now you'll be paying 6¢ 8,900 times. That is, the tax is:

$$\$0.06 \times \$8,900 = \$534$$

1% of \$8,900 is \$89. This is how much additional tax you pay in Example 24 over the tax you'd pay in Example 23. As a check notice that \$534 - \$445 = \$89

More generally, relative to Examples 23 and 24, every time the tax increases by 1% the total cost of the car increases by \$89. If the price of the car had been \$10,000 each increase of 1% would cost you \$100.

Sometimes we have a receipt that gives us a total price including the tax of a purchase. From this we might want to know how much the tax itself was. This is particularly important if we file the long form for income taxes. Since you do not pay a tax on taxes, all money you spent for taxes can be deducted from your income tax return. Let's see how this works in a particular situation.

Example 25

You find a receipt for a car that you purchased. The total cost, including a 5% sales tax, is $9,870. What was the price of the car before the sales tax was added on?

Answer: $9,400

The "program" is:

Marked Price X 1.05 = Total Price.

In this example, the $9,870 refers to *the total price*, not the marked price.

Therefore the "program" becomes:

Marked Price X 1.05 = $9,870

or

____________ X 1.05 = $9,870

Thus we must divide $9,870 by 1.05 and this gives us $9,400,

SPECIAL CAUTION

Do not deduct 5% of $9,870. $9,870 already includes the sales tax and you didn't pay a sales tax on the sales tax.

Get the point? If the marked price had been $9,870 then the formula would give: $9,870 X 1.05 = ________ However it was the total cost that was $9,870, so now the formula becomes: ______ X 1.05 = $9,870

Check

5% of $9,400 =	*$470*
Marked Price =	*$9400*
Total Price =	*$9870*

So for tax purposes, you can conclude that $470 of $9,870 went to cover the sales tax. This $470 can be itemized as an expense.

If you took 5% of $9,870 you'd get $493.50. Subtracting this from $9,870 you'd get $9,006.50 but this answer won't check. So remember; when you work with the total cost, the tax is already included--and we do not pay tax on the tax.

Let's conclude this module with another important example of constant rates that are useful to all consumers. This time let's talk about discounts that are offered on sales.

Actually the number of applications of constant rates is endless. However, once you understand what happens in a few examples, you can generalize the results to other applications.

We often hear an advertisement offering "40% off" What does this mean? If the ad refers to 40% off the marked price, we are being told that for every dollar the object is marked, we are going to save 40¢ (that is, they're taking off 40¢ per 100¢). In

this case we pay only \$0.60 for each dollar the object is marked. In terms of a formula:

Regular Price X 0.60 = Sale Price

That is, 60% of the regular price = sale price. In terms of a chart.

Make sure you see the difference between "off" and "of". If 40% is taken off the regular price, you're still paying 60% of the regular price.

Regular Price	*Sales Price*
\$1	*\$0.60*
\$2	*\$1.20*
\$3	*\$1.80*
\$4	*\$2.40*
(Multiples of \$1.00)	*(Multiples of \$0.60)*

So in terms of constant rate we're paying 60¢ per each regularly-priced dollar.

Example 26

At 40% off the regular price a radio costs \$84. What is the regular price of the radio?

Answer: \$140

Hint: The regular price has to be more than the sales price. So don't take 40% of \$84, which is less than \$84.

Reading the problem correctly is very important. Notice that the \$84 modifies the sales price. So in this case the regular price is the unknown. That is,

Regular Price X 0.60 = Sales Price

becomes:

__________ X 0.60 = \$84

Hence we divide \$84 by 0.60 to get:

$$\frac{\$84.00}{0.60} = \frac{\$8,400}{60}$$

```
     $ 1 4 0
60 )$8,4 0 0
   - 6 0
     2 4 0
   - 2 4 0
         0
```

Check: *40% of \$140 = \$56 = savings*
60% of \$140 = \$84 = sale price

Think of the rectangle as representing the regular price (100%). Then the situation is:

Amt you save	*Amt you pay*	
40%	*60%*	= 10
or \$56	*or \$84*	= \$14

If 60% of a number is \$84, 1% is 1/60 of \$84 or \$1.40. Hence 100% of the number is \$140, and 40% of the number is \$1.40 X 40 or \$56.

The importance of reading comprehension cannot be overemphasized.

Example 27

The regular price of a radio is $84. At 40% off the regular price, what would the sales price be?

Answer: $50.40

The formula is still given by

Regular Price X 0.60 = Sales Price

But now it is the Regular Price that gets replaced by $84 and the Sales Price that gets replaced by the blank. That is:

$84 X 0.60 = ______

and since $84 X 0.60 = $50.40, the sales price is $50.40. Your savings is $33.60.

If it helps, think in terms of a picture. This time the $84 represents the value of the entire rectangle:

Regular Price = $84

Amt you save	*Amt you pay*
40% of $84	*60% of $84*
$33.60	*$50.40*

Example 28

At 40% off the regular price of a radio you save $84. What was the regular price of the radio?

Answer: $210

This is tricky. This time the $84 is the savings, not a price you pay. The 40% off represents the savings. That is:

40% of the Regular Price = Savings

Since the savings is $84 we have that

40% of ______ = $84, or

0.4 X _____ = $84, or

______ = $84 ÷ 0.4 = $210.

Check: 60% of $210 = $126 (Sale Price)
40% of $210 = 84 (Savings)
100% of $210 = $210

60% of the Regular Price is the sales price. The sales price plus the savings is the regular price.

Again in terms of a rectangle:

Amt you save	*Amt you pay*
40%	*60%*
$84	*$126*

If 40% is $84, then 1% is 1/40 of $84 or $2.10 Therefore the regular price (100%) is $2.10 and 60% is $126.

Comparing Examples 26, 27 and 28, we see that no formula can help us if we can't describe what's given and what's being asked for. A calculator or computer will make the arithmetic easier, but without a *number sense* we might not know the correct arithmetic that has to be done. In all three problems we used the diagram:

In other words, if we don't understand what's happening and we have a calculator, we can get the right answer but to the wrong problem!

Regular Price

Amt you save	*Amt you pay*
40%	*60%*

But it takes reading comprehension to know how to fill in the rest of the numbers.

but each time the $84 stood for a different part of the problem.

Key Point

We can use calculators and computers to simplify the arithmetic, but there is no substitute for reading with understanding. Learn to read every word-problem carefully, and as a final precaution, always check your answers.

This concludes our introduction to constant rates. In the next module we'll apply our results to different measuring systems with special emphasis on the metric system.